F-SIGNATURE OF GRADED GORENSTEIN RINGS

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ABSTRACT. For a commutative ring R, the F-signature was defined by Huneke and Leuschke [H-L]. It is an invariant that measures the order of the rank of the free direct summand of $R^{(e)}$. Here, $R^{(e)}$ is R itself, regarded as an R-module through e-times Frobenius action F^e.

In this paper, we show a connection of the F-signature of a graded ring with other invariants. More precisely, for a graded F-finite Gorenstein ring R of dimension d, we give an inequality among the F-signature $s(R)$, a-invariant $a(R)$ and Poincaré polynomial $P(R,t)$.

$$s(R) \leq \frac{(-a(R))^d}{2^{d-1}d!} \lim_{t\to 1}(1-t)^d P(R,t)$$

Moreover, we show that $R^{(e)}$ has only one free direct summand for any e, if and only if R is F-pure and $a(R) = 0$. This gives a characterization of such rings.

1. Introduction

Let R be a reduced local ring of dimension d containing a field of characteristic $p > 0$ with a perfect residue field. Let $R^{(e)}$ be R itself, regarded as an R-module through the e-times composition of Frobenius maps F^e. In the following, we often assume that $R^{(e)}$ is F-finite. This is equivalent to $R^{(1)}$ being a finite R-module. The structure of $R^{(e)}$ has a close relationship with the singularity, multiplicity and Hilbert-Kunz multiplicity of R. For example, if R is a regular local ring of dimension d, then by a theorem of Kunz, $R^{(e)}$ becomes a free module of rank p^{ed}. The converse also holds, and thus, this property characterizes regular local rings.

In [H-L],Huneke and Leuschke investigated how many free direct summands are contained in $R^{(e)}$. Namely, they decomposed $R^{(e)}$ as $R^{(e)} = R^{a_q} \oplus M_q$ and examined a_q, where M_q is an R-module that does not contain any free direct summand. The number a_q is known to be independent of the decomposition. In this article, first we consider the case where $a_q = 1$ for any $q = p^e$.

We obtained the following result.

Theorem 1.1. *For any reduced graded F-finite Gorenstein ring R such that R has an isolated singularity and R_0 is a perfect field, the following are equivalent.*

(1) *$R^{(e)}$ has only one free direct summand, for each positive integer e.*
(2) *R is F-pure and $a(R) = 0$.*

Definition 1.2. Decompose $R^{(e)}$ as $R^{(e)} = R^{a_q} \oplus M_q$, where M_q is an R-module that does not contain R as a direct summand. If the limit exists, we set

$$s(R) = \lim_{q\to\infty} \frac{a_q}{q^d},$$

and call it *F-signature*.

This invariant was introduced by Huneke and Leuschke, and has been investigated by several researchers. In their paper [H-L], the following theorems were shown.

Theorem 1.3. *Let R be an F-finite reduced Cohen-Macaulay local ring with a perfect residue field and whose F-signature $s(R)$ exists. Then the following are equivalent.*

(1) $s(R) = 1$.
(2) *R is a regular local ring.*

Theorem 1.4. *Let R be an F-finite reduced complete Gorenstein local ring with a perfect residue field. Then the following are equivalent.*

(1) $s(R) > 0$.
(2) *R is F-rational.*

From these results, we can say that the F-signature contains enough information to characterize some kind of classes of commutative rings. Theorem 1.3 was generalized by Y. Yao to that $s(R) \geq 1 - \frac{1}{d!p^d}$ implies R is regular (see [Y]). Besides, as in the assumption of Theorem 1.3, it is also a nontrivial question whether the F-signature exists or not. Several papers have dealt with this problem; for example, Gorenstein local rings are known to admit F-signatures.

In this article, we obtained the following result.

Theorem 1.5. *For any reduced graded F-finite Gorenstein ring R and if R_0 is assumed to be a perfect field, we have the following inequality.*

$$s(R) \leq \frac{(-a)^d e'}{2^{d-1} d!}. \tag{1.1}$$

Here, d is the dimension of R, $a = a(R)$ is the a-invariant of R, and $e' = \lim_{t\to 1}(1-t^d)P(R,t)$, where $P(R,t)$ is the Poincaré series of R.

In dimension 2, the equality holds in (4.1) if R is regular or a rational double point. This equality never holds for regular rings of dimension greater than or equal to 3.

2. Preliminaries

In the following, we write $q = p^e$ for any positive integer e.

Proposition 2.1. *Let R be a d-dimensional Cohen-Macaulay local ring of characteristic p. The following are equivalent.*

(1) *R is F-rational.*
(2) *0 is tightly closed in $H^d_{\mathfrak{m}}(R)$.*

Proof. See [Sm], [H, Theorem 4.4]. □

Proposition 2.2. *For any F-pure Cohen-Macaulay positively graded ring R of dimension d and characteristic p, we have $a(R) \leq 0$.*

Proof. See [F-W], Remark 1.6. □

Proposition 2.3. *Let R be an F-injective Cohen-Macaulay graded ring of dimension d and characteristic p, which is an isolated singularity. If $a(R) < 0$, then R is F-rational.*

Proof. Let $\tau(R)$ be the test ideal of R. Because R has an isolated singularity, $\tau(R)$ is $\mathfrak{m} = \oplus_{n>0} R_n$-primary. Hence the annihilator of $\tau(R)$ in $H^d_{\mathfrak{m}}(R)$ is finitely generated.

If R is not F-rational, then there exists a non-zero element x in $H^d_{\mathfrak{m}}(R)$ that satisfies $cx^q = 0$ for all $c \in \tau(R)$ and q. Since $a(R) < 0$ and R is F-injective, $x^q \neq 0$ and $\deg x^q$ gets arbitrary small. Hence $\tau(R)x^q$ cannot be 0. A contradiction! □

3. F-signature

Let R be a reduced commutative ring of positive characteristic with a perfect residue field. As before, let p be char$R$ and let $q = p^e$ for any positive integer e.

In the following, we assume R is F-finite, i.e., $R^{(1)}$ is finite as an R-module.

Definition 3.1. Decompose $R^{(e)}$ as $R^{(e)} = R^{a_q} \oplus M_q$, where M_q is an R-module which does not contain R as a direct summand. If the limit exists, we set

$$s(R) = \lim_{q \to \infty} \frac{a_q}{q^d},$$

and call it *F-signature.*

Remark 3.2. For each e, we have $a_q \leq \mathrm{rank} R^{(e)} \leq q^d$. Thus $s(R)$ satisfies $s(R) \leq 1$.

Example 3.3. Let R be a regular local ring of dimension d with a perfect residue field. By the theorem of Kunz [Ku], we have $R^{(e)} = R^{q^d}$. Thus we have $a_q = q^d$, and $s(R) = 1$.

The converse also holds. See [H-L, Corollary 16].

Proposition 3.4. *Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Then we have $s(R) = s(\hat{R})$.*

Proof. This is trivial, because the completion is fully faithful. □

Theorem 3.5. *Let R be a Gorenstein ring. The following are equivalent.*

(1) $s(R) > 0$.
(2) *R is F-rational.*
(3) *R is F-regular.*

Proof. See [H-L, Theorem 11]. □

Example 3.6. For a 2-dimensional rational double point, we can calculate its F-signature several ways. See, for example, [W-Y]. The results are listed below.

type	equation	char$R$	$s(R)$
(A_n)	$f = x^2 + y^2 + z^{n+1}$	$p \geq 2$	$\frac{1}{n+1}$
(D_n)	$f = x^2 + yz^2 + y^{n-1}$	$p \geq 3$	$\frac{1}{4(n-2)}$
(E_6)	$f = x^2 + y^3 + z^4$	$p \geq 5$	$\frac{1}{24}$
(E_7)	$f = x^2 + y^3 + yz^3$	$p \geq 5$	$\frac{1}{48}$
(E_8)	$f = x^2 + y^3 + z^5$	$p \geq 7$	$\frac{1}{120}$

4. Proof of the theorem

In this section, we give a proof of the main theorem and calculate some examples. We use the notation of the previous section. Throughout this section, let $R = \oplus_{n \geq 0} R_n$ be an F-finite graded ring of positive characteristic. We denote $\mathfrak{m} = R_+ = \oplus_{n>0} R_n$.

Lemma 4.1. *Let R be a graded Gorenstein ring, and let $a = a(R)$. Then, for any homogeneous element α in R such that αF^e is split mono, we have*

$$\deg \alpha \leq -a(q-1).$$

Proof. Let $f_1, f_2, \ldots, f_d$ be a homogeneous system of parameter R, and let $b_i = \deg f_i$. Because αF^e is split mono, by tensoring $R^q/(f_1^q, f_2^q, \ldots, f_d^q)$ over R^q, we see that $\alpha R^q/(f_1^q, f_2^q, \ldots, f_d^q)$ is a free direct summand of $R/(f_1^q, f_2^q, \ldots, f_d^q)$.

If we take a generator z of $\mathrm{soc}(R/(f_1, f_2, \ldots, f_d))$, then it satisfies

$$\deg z = a + b_1 + b_2 + \cdots + b_d.$$

Because $\alpha R^q/(f_1^q, f_2^q, \ldots, f_d^q)$ is a free direct summand of $R/(f_1^q, f_2^q, \ldots, f_d^q)$, we have $\alpha z^q \neq 0$ in $R/(f_1^q, f_2^q, \ldots, f_d^q)$, and thus

$$\deg \alpha z^q \leq a(R/(f_1^q, f_2^q, \ldots, f_d^q)).$$

Because $a(R/(f_1^q, f_2^q, \ldots, f_d^q)) = a + q(b_1 + b_2 + \cdots + b_d)$, we obtain

$$\deg \alpha + q(a + b_1 + b_2 + \cdots + b_d) \leq a + q(b_1 + b_2 + \cdots + b_d).$$

Thus Lemma 4.1 follows. □

Theorem 4.2. *Let R be a reduced graded F-finite Gorenstein ring with an isolated singularity, and let us assume that $R_0 = k$ is a perfect field. The following are equivalent.*

(1) *$R^{(e)}$ has only one free direct summand for each positive integer e.*
(2) *R is F-pure and $a(R) = 0$.*

Proof. Because the implication from (2) to (1) is obvious by Lemma 4.1, it suffices to show the converse.

Suppose we have $a_q = 1$ for some $e \gg 1$. This means there exists an α such that αR is a free direct summand of $R^{(e)}$. α is not contained in R^0. Therefore, R is F-pure(see [HH, page 128,remark(c)]), which implies that $a(R) \leq 0$. If we suppose that $a(R) < 0$, then R becomes F-rational and $s(R) > 0$, contradicting $a_q = 1$. □

Theorem 4.3. *For any reduced graded F-finite Gorenstein ring R and if we assume that $R_0 = k$ is a perfect field, we have the following inequality.*

$$s(R) \leq \frac{(-a)^d e'}{2^{d-1} d!}. \tag{4.1}$$

Here, d is the dimension of R, a is the a-invariant, and $e' = \lim_{t \to 1}(1-t)^d P(R,t)$, where $P(R,t)$ is the Poincaré series of R.

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_{a_q}$ be a basis of the free direct summand of $R^{(e)}$. We assume that each α_i is a homogeneous element. Let $f_1, f_2, \ldots, f_d$ be homogeneous parameter systems of R, and let $b_i = \deg f_i$. We may assume that each b_i is large enough with respect to a_q and q. Take a generator z of $\mathrm{soc}(R/(f_1, f_2, \ldots, f_d))$. By Lemma 4.1, we have $\deg \alpha_i \leq -a(q-1)$. Because $\alpha_1 R^q \oplus \alpha_2 R^q \oplus \cdots \oplus \alpha_{a_q} R^q$ is a free direct summand of $R^{(e)}$,

$$\alpha_1 R^q/(f_1, f_2 \ldots, f_d) \oplus \alpha_1 R^q/(f_1, f_2 \ldots, f_d) \oplus \cdots \oplus \alpha_{a_q} R^q/(f_1, f_2 \ldots, f_d)$$

becomes a free direct summand of $R^{(e)}/(f_1, f_2 \ldots, f_d) R^{(e)} = R/(f_1^q, \ldots, f_d^q)R$.

Define a_q^- and a_q^+ by

$$\begin{aligned} a_q^- &= \{i \mid \deg \alpha_i < \frac{-a(q-1)}{2}\}, \\ a_q^+ &= \{i \mid \deg \alpha_i \geq \frac{-a(q-1)}{2}\}. \end{aligned}$$

And let r_n=$\dim_k(R/(f_1^q, \ldots, f_d^q)R)_n$. Because α_i's are linearly independent over k, we have

$$a_q^- \leq \sum_{n=0}^{\frac{-a(q-1)}{2}-1} r_n.$$

Similarly, because $\alpha_i z^q$'s are linearly independent over k, we have

$$a_q^+ \leq \sum_{n=\frac{-a(q-1)}{2}+q(a+b_1+\cdots+b_d)}^{-a(q-1)+q(a+b_1+\cdots+b_d)} r_n.$$

By the duality of the Gorenstein ring $R^{(e)}/(f_1, \ldots, f_d)R^{(e)} = R/(f_1^q, \ldots, f_d^q)R$, we obtain

$$r_{\frac{-a(q-1)}{2}+q(a+b_1+\cdots+b_d)+i} = r_{\frac{-a(q-1)}{2}-i}$$

and thus

$$a_q^+ \leq \sum_{n=0}^{\frac{-a(q-1)}{2}} r_n.$$

Thus we obtain

$$a_q = a_q^+ + a_q^- \leq 2 \sum_{n=0}^{\frac{-a(q-1)}{2}} r_n.$$

Dividing his by q^d and taking the limit, we obtain s(R) from the left-hand side. For large q,

$$2\sum_{n=0}^{\frac{-a(q-1)}{2}} r_n = 2\frac{e'}{d!}(\frac{-a(q-1)}{2})^d + (terms of lower degree) = \frac{(-a)^d e'}{2^{d-1} d!} q^d + (terms of lower degree).$$

Hence the result follows. □

Example 4.4. If $R = k[X_1, \ldots, X_d]$ is a polynomial ring over perfect field k, because $a(R) = -d$ and $e' = 1$, we have

$$\frac{(-a)^d e'}{2^{(d-1)d!}} = \frac{d^d}{2^{(d-1)}d!}.$$

The right hand side is equal to 1 for $d = 1, 2$ and greater than 1 for $d \geq 3$. Thus we have equality in (4.1) if and only if $d = 1, 2$.

Example 4.5. (4.1) becomes an equality if R is a 2-dimensional rational double point.

To confirm this, we introduce the following lemma.

Lemma 4.6. *Let R be a graded ring, and let x be a regular homogeneous element in degree b. Then*

$$(1 - t^b)P(R, t) = P(R/(x), t) \tag{4.2}$$

Proof. Since x is a homogeneous regular element of degree b,

$$0 \to R(-b) \xrightarrow{x} R \to R/(x) \to 0 \tag{4.3}$$

is exact. By the additivity of the dimension of k-vector spaces, we obtain

$$H(R/(x), n) = H(R, n) - H(R, n-b),$$

and thus

$$\begin{aligned} H(R/(x), n)t^n &= H(R, n)t^n - H(R, n-b)t^n, \\ H(R/(x), n)t^n &= H(R, n)t^n - t^b H(R, n-b)t^{n-b}. \end{aligned}$$

Here, $H(R, n)$ denotes the k-dimension of the degree n part of R. Taking the sum of these equations, we obtain (4.2). □

Lemma 4.7. *Let R be a graded Cohen-Macaulay ring, and let x be a homogeneous regular element of degree b. Then we have*

$$a(R/(x)) = a(R) + b.$$

Proof. See [G-W1](2.2.10) or [B-H](3.6.14). □

In the following, we demonstrate how to calculate the right-hand side of (4.1) in Theorem 4.3 for singularities of type A_n by using these lemmas. In this case, $R = k[x, y, z]/x^2 + y^2 + z^{n+1}$. We remark that because R is a hypersurface singularity, it is a complete intersection, in particular Cohen-Macaulay.

R can be regarded as a graded ring with $\deg x = \deg y = n+1, \deg z = 2$. With this grading, $x^2 + y^2 + z^{n+1}$ is a homogeneous element of degree $2(n+1)$, and thus we have

$$a(R) = -(n+1+n+1+2) + 2(n+1) = -2$$

by Lemma 4.7.

Next we calculate the Poincaré series. By Lemma 4.6, we have

$$(1 - t^{n+1})P(R, t) = P(R/(x), t).$$

Moreover, because y is a regular element in $R/(x)$ of degree $n+1$, we have

$$(1 - t^{n+1})P(R/(y), t) = P(R/(x, y), t)$$

again by Lemma 4.6. Since z is an element of degree 2 in $R/(x, y) = k[x, y, z]/z^{n+1}$, we have

$$P(R/(x, y), t) = 1 + t^2 + \cdots + t^{2n} = \frac{1 - t^{2(n+1)}}{1 - t^2}.$$

Combining with the above two equations, we obtain

$$P(R, t) = \frac{1 - t^{2(n+1)}}{(1 - t^{n+1})(1 - t^{n+1})(1 - t^2)},$$

and thus $e' = \frac{1}{n+1}$. The right-hand side of (4.1) can be calculated to be $\frac{1}{n+1}$, and thus is equal to e' in this case.

The following is the list of a and e' calculated by the above method.

type	e'	$a(R)$	RHS of (4.1)
(A_n)	$\frac{1}{n+1}$	-2	$\frac{1}{n+1}$
(D_n)	$\frac{1}{n-1}$	-1	$\frac{1}{4(n-2)}$
(E_6)	$\frac{1}{6}$	-1	$\frac{1}{24}$
(E_7)	$\frac{1}{12}$	-1	$\frac{1}{48}$
(E_8)	$\frac{1}{30}$	-1	$\frac{1}{120}$

5. Local case

Lastly, we give a local version of theorem 4.2 .

Theorem 5.1. *Let $(R, \mathfrak{m})$ be a F-finite Gorenstein local ring that is F-rational on a punctured spectrum and assume that the residue field is perfect. The following are equivalent.*

(1) *$R^{(e)}$ has only one free direct summand, for each positive integer e.*
(2) *R is F-pure and not F-rational*

Proof. The implication from (1) to (2) is the same as the graded case. To show the converse, we use the splitting prime. For the definition and the behavior, see [A-E2]. Let $\mathfrak{p}$ be the splitting prime. Because R is F-pure, $\mathfrak{p}$ is not unit ideal. Because $\mathfrak{p}$ contains the test ideal, $\mathfrak{p}$ equals to $\mathfrak{m}$.(See [A-E2, remark3.5.].)
But then $a_q = 1$ by the remark again, namely $R^{(e)}$ has only one free direct summand, for each positive integer e.

□

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